Robust adaptive finite element schemes for nonlinear viscoelastic solid deformation: an investigative study

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1 Phase I research

This project is concerned with developing robust numerical schemes for the adaptive solution of quasistatic viscoelasticity problems. Specifically, we are investigating and comparing the two models resulting from **hereditary integral** and **internal variable** formulations with regard to: mathematical feasibility; computational efficiency; flexibility; and, reliability. We refer to the original *Seed Project Proposal* for further details, and in particular to Section 1 where the error estimate is described in the context of the hereditary integral formulation of a model problem posed in one space dimension plus time.

In this first phase of the research we have addressed the first of the objectives set out in the proposal:

> *To extend the one-dimensional* a posteriori *error analysis and adaptive scheme to linear quasistatic problems in higher dimensions.*

Apart from a few minor technical details the theoretical aspect of this work is now substantially complete in the following sense.

We have obtained ***a posteriori* error estimates** for a space-time finite element discretization of the problem wherein the displacement is approximated by a finite element solution that is piecewise linear and continuous in the space variables, and either piecewise constant or linear, and discontinuous, in the time variable. These discontinuities occur at the discrete time levels $t_1, t_2, \ldots$, and allow the space-mesh to change (be adapted) in time.

In terms of spatial discretization our estimates measure the displacement errors in the elastic-energy norm. This is usual for the closely related linear elasticity problem. For the time discretization we measure this error in either the maximum energy norm over the time interval, or in a norm especially designed for the problem in order to overcome the difficulty presented by the lack of strong temporal stability of the solution. To summarize our results we refer to items in the draft BICOM technical report:

> **Space-time finite element method with a-posteriori Galerkin energy-error estimates for linear quasistatic viscoelasticity problems (draft)**

in the following way. We refer to "item" in this report by using the notation $\langle item \rangle$. For example, to refer to the equation labelled "6" in the report we use $\langle Equation\ 6 \rangle$.

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Some notation: we use $\boldsymbol{u}$ and $\boldsymbol{U}$ to denote respectively the exact and finite element displacements vectors (these are functions of space, $\boldsymbol{x}$, and time, t), and denote the error as $\boldsymbol{e} = \boldsymbol{u} - \boldsymbol{U}$. The energy norm of the error is given by $\|\boldsymbol{e}\|$ and—as usual—we use h to indicate the space-mesh size, and k for the time steps. The problem is considered as posed in space in a domain Ω, and in time during the time interval $\mathcal{J} = [0, T]$.

The basic *a posteriori* error estimate in ⟨*Theorem 13*⟩ is:

$$\max_{1\leq q\leq p} \|\boldsymbol{e}(t_q)\| \leq S(t_p)\Big(\mathcal{E}_\Omega(t_p;\boldsymbol{U}) + \mathcal{E}_\mathcal{J}(t_p;\boldsymbol{U})\Big),$$

for every time level $t_p = t_1, t_2, t_3, \ldots$. Here $S(t)$ is a stability factor which is known precisely in terms of the **fading memory** of the viscoelasticity problem, and $\mathcal{E}_\Omega$, $\mathcal{E}_\mathcal{J}$ are residual terms that are **computable** in terms of the numerical solution $\boldsymbol{U}$ and the applied loads. These terms $\mathcal{E}_\Omega$, $\mathcal{E}_\mathcal{J}$ are supposed to measure respectively the errors due to space and time discretization.

In fact $\mathcal{E}_\Omega$ can be used to estimate the "space errors" and also to design the space mesh in order to achieve **space-error control for a user specified tolerance level.** But, in its basic form $\mathcal{E}_\mathcal{J}$ is unstable ⟨*Section 7*⟩, and therefore of no value. This is a direct consequence of the lack of strong temporal stability in the underlying problem.

The term $\mathcal{E}_\mathcal{J}$ can be stabilized at the price of measuring the residual term it contains in a weaker norm ⟨*Theorem 16*⟩. This increases the computational "cost" of the algorithm.

To address this difficulty we have also produced in ⟨*Theorems 24 and 27*⟩ *a posteriori* estimates for the error measured in a weaker norm especially tailored to the problem. This time we have

$$\operatorname*{weak\,max}_{1\leq q\leq p} \|\boldsymbol{e}(t_q)\| \leq S(t_p)\Big(\mathcal{E}^*_\Omega(t_p;\boldsymbol{U}) + \mathcal{E}^*_\mathcal{J}(t_p;\boldsymbol{U})\Big).$$

The definition of the "weak max" norm (denoted by $\|\cdot\|_{W_*^{-1}(0,t_p;\text{energy})}$ in the draft technical report) is loosely based on the dual norm $\|\cdot\|_{W_\infty^{-1}(0,t_p)}$ as used in Sobolev space theory. We refer to ⟨*Section 8*⟩ for further details.

In this latter estimate the term $\mathcal{E}^*_\Omega(t_p;\boldsymbol{U})$ is substantially the same as $\mathcal{E}_\Omega(t_p;\boldsymbol{U})$, but now $\mathcal{E}^*_\mathcal{J}(t_p;\boldsymbol{U})$ allows adaptive time step selection to **control the "time errors" for a user-specified tolerance level** also. Thus the algorithm will be able to **automatically** adjust the space-time discretization so as to produce an approximate solution $\boldsymbol{U}$ for which,

$$\operatorname*{weak\,max}_{1\leq q\leq p} \|\boldsymbol{u}(t_q) - \boldsymbol{U}(t_q)\| \leq \text{TOL} = \textbf{ user-specified} \text{ tolerance.}$$

The price of this is error control in a less physical norm.

We have also derived ***a priori* error estimates** for the scheme in ⟨*Theorem 33*⟩, and have shown **reliability** of the *a posteriori* error estimates in ⟨*Lemmas 35, 36 and 37*⟩.

We expect at least two research papers to arise from this work, dealing with the *a priori* and *a posteriori* error analysis respectively.

2 Future work

The next phase of the research will address Objective 2 in the Seed Project Proposal, which concerns the possibility of deriving adaptive finite element methods for the **internal variable** formulation of the problem. The timing here is deliberate so as to optimize the collaboration with Dr. Arthur Johnson, who plans to visit BICOM during the period March 1–8, 1998. When we have reached a suitable point in this work we will develop software to implement our results for Objectives 1 and 2 before moving on to address Objectives 3 and 4.

3 Administrative Actions and Other Details

- During this first phase Dr. S. Shaw has been supported in part from this contract, and will continue to be supported in part for the duration.
- Dr. S. Shaw presented a lecture on our results on adaptive methods for viscoelastic solid deformation at Imperial College, London, in November 1997.
- Professor Whiteman presented the models and *a posteriori* error estimates in seminars at Texas A and M University, and at the University of Texas at Austin in November and December 1997.
- The first receipt of funds from this contract will result from the submission of this first interim report.